Rainbow Total Connection Number of Sun Graphs and Sunlet Graphs

Jonah Gay V. Pedraza
Samar State University

Abstract

A path P connecting two vertices u and v in a totally colored graph G is called a rainbow total-path connecting u and v if all elements in $V(P) \cup E(P)$, except for u and v, are assigned distinct colors. A total-colored graph G is rainbow total-connected if for every two vertices of G there is a rainbow total-path connecting the two. The rainbow total-connection number of a graph G is the minimum colors such that G is rainbow total-connected. In this paper, we gave the rainbow totalconnection number of sunlet graphs, and a tight upper bound of the rainbow total-connection number of sunlet graphs.

Keywords: total-colored graph; rainbow total-connection number; sun graph; sunlet graphs.

Introduction

Chartrand et al. (2008) introduced the concept rainbow coloring. They determined rainbow connection number of the cycle, path, tree and wheel graphs. Since then many are studying the concept. Please see Li et al. (2013) and Sun et al. (2012). Li et al. (2013) studied the rainbow connection numbers of line graphs in the light of particular properties of line graphs and gave two sharp upper bounds for rainbow connection number of a line graph. While, Sun et al. (2012) investigated the rainbow connection number of the line graph, middle graph and total graph of a connected trianglefree graph and obtained three (near) sharp upper bounds in terms of the number of vertex-disjoint cycles of the original graph. Continuing the study of rainbow coloring, Uchizawa et al. (2011) introduced and studied the rainbow totalconnection number of graphs. Later, Sun (2013, 2015) also studied rainbow total-connection number. They characterized the rainbow total-connection number of trees, and gave the rainbow total-connection number of cycles, path and wheels. In this paper, we gave the rainbow totalconnection number of some wheel related graphs. In particular, we gave the rainbow total-connection number of sunflower graphs, lotus inside circle and helms.

A graph G is an ordered pair (V,E) where V is a non-empty finite set and E is a family of two element subsets of V. The elements of V are called *vertices* and the elements of E are called *edges*. If $\{u,v\}$ is an edge, then we say that vertices u and v are *adjacent*, and that u and v are *incident* to $\{u,v\}$. We write edge $\{u,v\}$ concisely as uv . The *path* $P_n = (v_1, v_2, \ldots, v_n)$ is the graph with vertices $v_1, v_2, \ldots, v_n)$ and edges $v_1, v_2, v_2\ v_3, \ldots, v_{n-1}\ v_n$.

The *cycle* $C_n=[v_1,v_2,\ldots,v_n]$ is the graph with distinct vertices $v_1,v_2,\ldots,v_n$ and edges $v_1v_2, v_2v_3,\ldots,v_{n-1}v_n, v_nv_1$. A *complete* graph K_n is the graph with n vertices and any two vertices is connected by an edge. The *complement* of a graph G, denoted by $\overline{G}$, is the graph with the same vertices as G and two vertices in $\overline{G}$ are adjacent if they are not adjacent in G.

A total coloring of a graph $G=(V,E)$ is a function f from $V\cup E$ to a set C whose elements are called *colors*. In this case, we say that G is totally colored. A path P connecting two vertices u and v in a totally colored graph G is called a rainbow total path between u and v if all the elements in $[V(P)\cup E(P)]\backslash\{u,v\}$ are assigned distinct colors. A totally colored graph is *rainbow total connected* if it has a rainbow total path in between every two vertices. (In this case, we say that f is a *rainbow total coloring* of G.) The rainbow total connection number of a graph G is the minimum number of colors such that G is rainbow total connected.

The *sun graph*, denoted by S_n, is the graph of order $2n$ obtained by adding vertices w_i joined by edges to vertices v_i and $v_{i+1(\mathrm{mod}\, n)}$ of the cycle $C_n=[w_1,w_2,\ldots,w_n]$ for every $i=1,2,\ldots,n$.

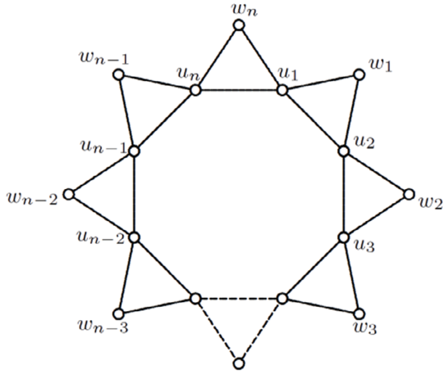

Figure 1. The sun graph by S_n.

The *sunlet graph* L_n of order $2n$ is the graph obtainedfrom the cycle $C_n=[w_1,w_2,\ldots,w_n]$ by attaching pendant edges v_iw_i for every $i=1,2,\ldots,n$.

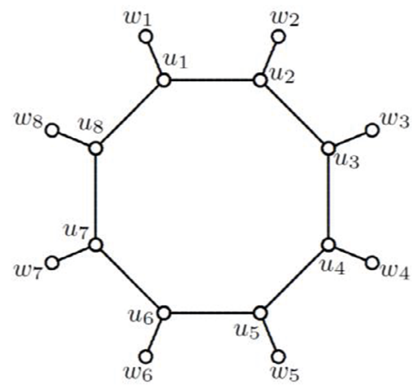

Figure 2. The sunlet graph by L_n.

Hereafter, please refer to Yellen et al. (2000) for concepts that are used but were not discussed in this paper.

Results

This section presents the results of this study.

Total Rainbow Connection Number of Sunlet Graphs

This subsection gives the rainbow total connection number of sunlet graph. Remark 1 states that the rainbow connection number of a connected graph G is greater than or equal twice its diameter minus 1.

Remark 2.1. *Let $G=(V,E)$ be a graph with diameter d. Then rtc* $(G)\geq 2d-1$.

To see this, let $u,v\in V$ such that the distance in between u and v, $d(u,v)$, is equal to d. Note that any rainbow path connecting u and v requires $2d–1$ colors. Hence, rtc (G) = $2d–1$.

The next result gives a tight lower bound of the rainbow connection number of sunlet graphs.

Lemma 2.2. *Let L_n be the sunlet graph of order $2n$. Then $rtc\ (L_n) \geq 2n$.*

Proof :
Let L_n be the sunlet graph of order $2n$ obtained from the cycle $C_n=[w_1,w_2,w_3,w_4]$ by attaching pendant edges v_iw_i for every $i=1,2,...,n$. Let f be a rainbow total coloring of L_n.

Claim 1. $f(w_i) \neq f(w_j)$ for $i \neq j$
Suppose that there exist i_0 , j_0 with $i_0 \neq j_0$ and $f\ (w_{i0}) = f\ (w_{j0})$. Then there is no rainbow total path connecting v_{i0} and v_{j0}. This is a contradiction since f is a rainbow total coloring of L_n. This shows the claim.

Claim 2. $f(v_iw_i) \neq f(v_jw_j)$ for $i \neq j$
Suppose that there exist i_0 , j_0 with $i_0 \neq j_0$ and $f\ (v_{i0}\ w_{i0}) = f\ (v_{j0}\ w_{j0})$. Then there is no rainbow total path connecting v_{i0} and v_{j0}. This is a contradiction since f is a rainbow total coloring of L_n. This shows the claim.

Claim 3. $f(v_iw_i) \neq f(w_j)$ for all i and j
Suppose that there exist i_0 , j_0 with $f\ (v_{i0}w_{i0}) = f(w_{j0})$. Then there is no rainbow total path connecting v_{i0} and v_{j0}. This is a contradiction since f is a rainbow total coloring of L_n. This shows the claim.

By Claim 1, we need n colors for $w_1,w_2,...,w_n$. By Claim 2, we also need n colors for $v_1,w_1,v_2\ w_2,...,\ v_n\ w_n$. By Claim 3, the colors for $w_1,w_2,...,w_n$ are distinct from $v_1,w_1,v_2\ w_2,...,\ v_n\ w_n$. Hence there should be at least $2n$ colors in the co-domain of f. Therefore, $rtc\ (L_n) \geq 2n$. **QED**

The lower bound stated in Lemma 2.2 attains equality when n is odd. The next theorem shows this.

Theorem 2.3. *Let L_n be the sunlet graph of order $2n$. if n is odd, then $rtc\ (L_n) = 2n$.*

Proof :
Let L_n be the sunlet graph of order $2n$ obtained from the cycle $C_n=[w_1,w_2,...,w_n]$ by attaching pendant edges v_iw_i for every $i=1,2,...,n$. By Lemma 2.2, $rtc\ (L_n) \geq 2n$. Let f be a total coloring of L_n given by

$$f(x) = \begin{cases} 2i-1 & , \text{ if } x = v_iw_i \\ 2i & , \text{ if } x = w_i \\ 1 & , \text{ if } x = v_i \text{ for all } i \\ f\left(v_{i+\lceil n/2 \rceil (\mathrm{mod}\, n)} w_{i+\lceil n/2 \rceil (\mathrm{mod}\, n)}\right) & , \text{ if } x = w_i w_{i+1 (\mathrm{mod}\, n)}. \end{cases}$$

If n is odd, then f is a rainbow total coloring of L_n using $2n$ colors. Hence, $rtc\ (L_n) \leq 2n$. Accordingly, $rtc\ (L_n) = 2n$. **QED**

For the even case, the equality may not hold as shown in the remark below.

Remark 2.4. *Let L_4 be the sunlet graph of order 8 . Then $rtc\ (L_4) > 8$.*

To see this, let L_4 be the sunlet graph of order 8 obtained from the cycle $C_n=[w_1,w_2,w_3,w_4]$ by attaching pendant edges v_iw_i for $i=1,2,3,4$. Let f_1 be a total coloring of L_n given by the following.

v	$f_1(v)$	e	$f_1(e)$
v_1	1	$w_1 w_2$	a
v_2	1	$w_2 w_3$	b
v_3	1	$w_3 w_4$	c
v_4	1	$w_4 w_1$	d
w_1	2	$u_1 w_1$	1
w_2	4	$u_2 w_2$	3
w_3	6	$u_3 w_3$	5
w_4	8	$u_4 w_4$	7

We observed that when each of a, b, c and d are assigned with any of the numbers 1, 2, 3, 4, 5, 6, 7 and 8, f is not a rainbow total coloring. Hence, $rtc(L_4)>8$. Thus, we state the following conjecture

Conjecture 2.4. *Let L_4 be the sunlet graph of order $2n$. If n is even number, then $2n < rtc(L_n)$.*

The next remark gives an upper bound of the rainbow total coloring number of sunlet graphs L_n.

Remark 2.4. *Let L_4 be the sunlet graph of order $2n$. Then $rtc(L_n) \leq 3n$.*

To see this, let L_n be the sunlet graph of order $2n$ obtained from the cycle $C_n=[w_1,w_2,...,w_n]$ by attaching pendant edges $v_i w_i$ for every $i=1,2,...,n$. Let f be a total coloring of L_n given by

$$f(x)=\begin{cases} 2i-1 & , \quad \text{if } x=v_i w_i \\ 2i & , \quad \text{if } x=w_i \\ 1 & , \quad \text{if } x=v_i \text{ for all } i \\ 2n+i & , \quad \text{if } x=w_i w_{i+1(\bmod n)} \end{cases} .$$

Then f is a rainbow total coloring of L_n using $3n$ colors. Hence, $rtc(L_n) \leq 3n$.

Total Rainbow Connection Number of Sun Graphs

This subsection gives a tight upper bound of the total rainbow connection number of sun graph. For brevity, we use the following notation.

Notation 3.1. Let A and B be sets with $A \subseteq B$. If $f: A \rightarrow \{1,2,...,c\}$ is a mapping, then we define $f' : B \rightarrow \{1,2,...,c\}$ as follows

$$f'(x)=\begin{cases} f(x) & , \quad \text{if } x \in A \\ 1 & , \quad \text{otherwise} \end{cases}$$

The next lemma gives a tight lower bound of the rainbow total coloring of sun graphs.

Lemma 3.1. *If H is a spanning subgraph of G, then $rtc(G) \leq rtc(H)$.*

Proof :

Let H be a spanning subgraph of G. Let $A=\{f': f$ is a rainbow total coloring of $H\}$ and $B=\{g : g$ is a rainbow total coloring of $G\}$. Since for every rainbow total coloring f of H, f' is a rainbow total coloring of G, $A \subseteq B$.

Let $C=\{|f(V(G) \cup E(G))| : f \in A\}$ and $D=\{|g(V(G) \cup E(G))|: g \in B\}$. Since $A \subseteq B$, $C \subseteq D$. Hence, min $D \leq$ min C , that is, $rtc(G) \leq rtc(H)$. **QED**

The next corollary gives a nearly tight upper bound of the rainbow total coloring number of sun graphs n S when n is odd.

Corollary 3.2. *Let S_n be a sun graph of order $2n$. If n is odd, then $rtc(S_n) \leq 2n$.*

Proof :
We note that L_n is a spanning subgraph of S_n. Hence, by Lemma 3.1 and Theorem 2.2, $rtc\ (S_n) \leq rtc\ (L_n) \leq 2n$. **QED**

The next remark shows that the bound presented in Corollary 3.2 is nearly tight.

Remark 3.3. *Let S_3 and S_4 be the sun graphs of order 3 and 4, respectively. Then*
1. $rtc\ (S_3) = 3$ *, and*
2. $rtc\ (S_4) = 5$*.*

To see this, let S_3 be the sun graph of order 6 obtained by adding vertices w_i joined by edges to vertices v_i and i $v_{i+1(\mathrm{mod}\,n)}$ of the cycle $C_3=[v_1,v_2,v_3]$ for every $i = 1,2,3$. Define $f_2{:}V(S_3) \cup E(S_3) \rightarrow \{1,2,3\}$ by the following.

Table 2. Images of the elements of $V(S_3) \cup E(S_3)$

v	$f_2(v)$	e	$f_2(e)$
v_1	2	$v_1 w_1$	3
v_2	2	$v_2 w_1$	1
v_3	2	$v_2 w_2$	3
w_1	2	$v_3 w_2$	1
w_2	2	$v_3 w_3$	3
w_3	2	$v_1 w_3$	1
		$v_1 v_2$	3
		$v_2 v_3$	3
		$v_3 v_1$	3

Then f_2 is a rainbow total coloring in S_3. Hence, $rtc\ (S_3) \leq 3$. By Remark 2.1, $rtc\ (S_3) \geq 2\ (2)\ –1=3$. Therefore, $rtc\ (S_3) = 3$.

Next, let S_4 be the sun graph of order 8 obtained byadding vertices w_i joined by edges to vertices v_i and $v_{i+1(\mathrm{mod}\,n)}$ of the cycle $C_4=[v_1,v_2,v_3,v_4]$ for every $i = 1,2,3,4$. Define $f_3{:}V(S_4) \cup E(S_4) \rightarrow \{1,2,3,4,5\}$ by the following.

Table 3. Images of the elements of $V(S_4) \cup E(S_4)$

v	$f_3(v)$	e	$f_3(e)$
v_1	4	$v_1 w_1$	5
v_2	4	$v_2 w_1$	1
v_4	4	$v_2 w_2$	5
v_3	2	$v_3 w_2$	1
w_1	4	$v_3 w_3$	5
w_2	4	$v_4 w_3$	1
w_3	2	$v_4 w_4$	5
w_4	4	$v_1 w_4$	2
		$v_1 v_2$	1
		$v_2 v_3$	3
		$v_3 v_4$	3
		$v_4 v_1$	1

Then f_3 is a rainbow total coloring in S_4. Hence, $rtc\ (S_4) \leq 5$. By Remark 2.1, $rtc\ (S_4) \geq 2\ (3)\ –1=5$. Therefore, $rtc\ (S_4) = 5$.

Conclusion

With the advent of new important results, this article is very interesting. The constructions of the different theorems were realized using the definition and properties of the concept *rainbow total connection number*. Also, other properties focusing on the two special cycle-related graphs, the sun graph and sunlet graph, were realized.

Moreover, the writer recommend that the rainbow total connection number of the other cycle related graphs mentioned in Ponraj et al. (2015)

be determined also.

Acknowledgement

The researcher would like to thank Samar State University for supporting this paper.

References

Yuefang S., On rainbow total-coloring of a graph, Discrete Applied Mathematics 5 (3) (2015) 260-276. http://dx.doi.org/10.1016/j.dam.2015.05.012.

Chakraborty S., Fischer E., Matsliah A. & Yuster R. (2009), Hardness and algorithms for rainbow connectivity, 26th International Symposium on Theoretical Aspects of Computer Science STACS 2009 (2009), 243-254.

Chartrand G., Johns G.I., Mc Keon K.A. & Zhang P. (2008), Rainbow connection in graphs. Math. Bohem. 133 (2008) 85-98.

Li X., Shi Y. & Sun Y. (2013), Rainbow connection in graphs – A survey, Graphs Combin. 29 (2013) 1-38.